GROUP THEORY 2024 - 25, EXERCISE SHEET 4

Exercise 1. *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. Let p be a prime number and G a group of order p^2. Let $Z(G)$ be the center of G. Using the following steps, show that if G is not cyclic, then it is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(1) Using the action of G on itself by conjugation, show that $Z(G) \neq \{0\}$ is non-trivial;
(2) Show that $G/Z(G)$ is cyclic and deduce that G is abelian;
(3) Show the result.

Exercise 3. Show that $\mathbb{Z}/4\mathbb{Z}$ cannot be written as the semi-direct product $\mathbb{Z}/2\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$ for any homomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/2\mathbb{Z})$.

Exercise 4. *Internal semi-direct product*

(1) **Definition:** Let G be a group and let $K, L \subseteq G$ be subgroups. We say that G is the internal semi-direct product of K with L if the following properties hold:

   (a) K is a normal subgroup of G.
   (b) $K \cap L = \{1\}$.
   (c) $KL = G$.

   Note that if G is an internal semi-direct product of K with L then since K is a normal subgroup of G, there is an action of L on K by automorphisms, namely $l \cdot k := lkl^{-1}$ for $l \in L$ and $k \in K$. Let φ denote the corresponding homomorphism $L \to \mathrm{Aut}(K)$. Show that

$$G \cong K \rtimes_{\varphi} L.$$

(2) Suppose furthermore that L is also a normal subgroup of G, show that $kl = lk$ for all $k \in K$ and $l \in L$. Observe that this implies that $\varphi : L \to Aut(K)$ is the trivial homomorphism. Conclude that

$$G \cong K \times L.$$

Exercise 5. Let $G = K \rtimes_{\psi} L$ for some groups K, L and a homomorphism $\psi : L \to Aut(K)$. Verify that G is the internal semi-direct product of $K \times \{1\}$ with $\{1\} \times L$ in G. Furthermore check that

$$(\psi_l(k), 1) = (1, l) \cdot (k, 1) \cdot (1, l)^{-1}$$

in G. Using this, conclude that

$$G = K \rtimes_\psi L \cong (K \times \{1\}) \rtimes_\varphi (\{1\} \times L).$$

Where $\varphi : \{1\} \times L \to Aut(K \times \{1\})$ corresponds to the conjugation action of $\{1\} \times L$ on $K \times \{1\}$ in G.

Exercise 6. Write S_3 as a semi direct product of subgroups.

Exercise 7. Let $n \geq 1$ be a positive integer.

(1) Find all possible homomorphisms $\varphi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/4\mathbb{Z})$;
(2) Describe their associated semi-direct product $\mathbb{Z}/4\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$;
(3) Is one of them isomorphic to D_8 ?
(4) Find a homomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/n\mathbb{Z})$ such that $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$.

Exercise 8. Let F be any field. The aim of this exercise is to show that

$$GL_n(F) \cong SL_n(F) \rtimes_\varphi F^\times$$

for some $\varphi : F^\times \to Aut(SL_n(F))$.

We will do this by showing that the following short exact sequence splits on the right (Refer to Proposition 10 of Lecture 4 of the notes):

$$1 \to SL_n(F) \xrightarrow{i} GL_n(F) \xrightarrow{\det} F^\times \to 1.$$

That is, construct a group homomorphism $\phi : F^\times \to GL_n(F)$ such that $\det \circ \phi = \mathrm{Id}_{F^\times}$. What is the map $\varphi : F^\times \to Aut(SL_n(F))$ which corresponds to the section ϕ that you have constructed such that $GL_n(F) \cong SL_n(F) \rtimes_\varphi F^\times$?